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TECHNICAL NOTE 3146

NOTE ON THE AERODYNAMIC HEATING OF AN OSCILLATING SURFACE

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SUMMARY

An analysis of the temperature distributions in a fluid over an oscillating surface with heat transfer is made and associated heat-transfer parameters are compared with those for the case of conduction at a stationary surface with the same initial temperature potential. It is found that the heat transfer for the oscillating surface can be considerably different from that for conduction alone. The effect of the surface oscillations on the thermal state of the fluid is studied by means of average static- or total-temperature defects, and it is demonstrated that the oscillations could alter the fluid temperature appreciably.

INTRODUCTION

The increased improvement of present-day propulsion systems and the development of new propulsion systems have posed numerous new problems in the field of heat transfer. Elucidation of unusually high heat-transfer coefficients which are apparently encountered in unsteady flows and means of increasing heat-transfer coefficients under given conditions are greatly desired. As a preliminary attempt to gain insight into such problems, it seems worth while to consider the heat-transfer aspects of the classical problem wherein the fluid motion is induced by oscillating a conducting surface axially in viscous fluid. In particular, the effect of disturbing the equilibrium (steady state) conditions after the periodic motion of the fluid has been established will be studied. The temperature distributions in the fluid are determined as exact closed-form solutions of the energy equation pertinent to the problem and, hence, related heat-transfer parameters can be compared with those for a stationary surface to demonstrate the effect of the surface oscillations. Other exact solutions of the energy equation for somewhat analogous problems are presented in references 1 to 3.

ANALYSIS

The laminar motion of an incompressible viscous fluid induced by the axial (longitudinal) oscillations of a bounding plane surface doubly infinite in extent is well known and is reported in numerous sources (see Schlichting, ref. 4, for example). The equations of motion for this case, assuming constant physical properties, reduce to the classical heat-conduction equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

where u is the velocity component parallel to the surface, t denotes the time, ν is the kinematic viscosity, and y is the coordinate normal to the surface. (All symbols are defined in the appendix.) The associated boundary conditions are

$$u(0,t) = U \cos nt \tag{2}$$

and

$$u(\infty,t) = 0 \tag{3}$$

where U denotes the amplitude and n the frequency of the oscillations. The velocity distribution, after sufficient time has elapsed for the periodic motion to be established, is given by

$$u(y,t) = U \exp\left(- \sqrt{n/2\nu}\, y\right) \cos\left(nt - \sqrt{n/2\nu}\, y\right) \tag{4}$$

The appropriate energy equation is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 \tag{5}$$

where T denotes static temperature, α is the thermal diffusivity, and c_p is the specific heat at constant pressure. Note that the last term on the right of equation (5) is due to aerodynamic heating and describes the increase in fluid temperature caused by the oscillations. In order for the problem to be physically reasonable, however, after a long period of time the system should be in a state of thermal equilibrium (that is, in essentially a steady state) and it is thus necessary to cool the surface. The thermal equilibrium state can be described by first solving equation (5) subject to the boundary conditions

$$T(0,t) = \overline{T}_w \tag{6}$$

and

$$T(\infty,t) = T_\infty \qquad (7)$$

that is, the oscillating surface is maintained at some uniform temperature $\overline{T}_w$. It is unnecessary here to specify any initial conditions, for they will not influence the results after a sufficiently long time.

After the explicit form of the nonhomogeneous or aerodynamic heating term in equation (5) is determined by means of equation (4), a particular solution of equation (5) was found to be

$$T_p = -\frac{U^2 Pr}{4c_p} \exp\left(-\sqrt{2n/\nu}\,y\right)\left[1 + \frac{\cos(2nt - \sqrt{2n/\nu}\,y)}{2 - Pr}\right] \qquad (8)$$

where $Pr = \nu/\alpha$ is the Prandtl number. The complementary function to be added to equation (8) in order that the complete solution satisfies the boundary conditions given by equations (6) and (7) is

$$T_c = \overline{T}_w + \frac{U^2 Pr}{4c_p}\left[1 + \frac{\exp(-\sqrt{n/\alpha}\,y)}{2 - Pr}\cos(2nt - \sqrt{n/\alpha}\,y) - \mathrm{erf}\left(\frac{y}{2\sqrt{\alpha t}}\right)\right]$$
$$+ (T_\infty - \overline{T}_w)\,\mathrm{erf}\left(\frac{y}{2\sqrt{\alpha t}}\right) \qquad (9)$$

where erf( ) is the error function and is defined as

$$\mathrm{erf}(\beta) \equiv \frac{2}{\sqrt{\pi}}\int_0^\beta \exp(-r^2)\,dr$$

so that

$$\mathrm{erf}(0) = 0$$

and

$$\mathrm{erf}(\infty) = 1$$

The complementary function is synthesized from well-known solutions of the heat equation; for example, the term containing the product of the exponential and trigonometric functions corresponds to the solution of the heat-conduction equation for the case where the temperature on a

stationary surface is specified to be a harmonic function of time (see
ref. 5, p. 47), and the two error-function terms combined correspond to
the solution of the problem of unsteady heat conduction to or from a
stationary surface due to a temperature difference or potential of
$T_\infty - \frac{U^2 Pr}{4c_p} - \overline{T}_w$. Note that the error-function terms describe the tran-
sient heat transfer which will vanish for large time. The temperature
distribution in the equilibrium state (that is, after a long time) should
be independent of time except for periodic terms and is given by the sum
of equations (8) and (9) with the error-function terms vanishing as

$$T_e = \overline{T}_w + \frac{U^2 Pr}{4c_p}\left\{1 - \exp\left(-\sqrt{\frac{2n}{\nu}}\,y\right) + \frac{1}{2-Pr}\left[\exp\left(-\sqrt{\frac{n}{\alpha}}\,y\right)\cos\left(2nt - \sqrt{\frac{n}{\alpha}}\,y\right)\right.\right.$$
$$\left.\left. - \exp\left(-\sqrt{\frac{2n}{\nu}}\,y\right)\cos\left(2nt - \sqrt{\frac{2n}{\nu}}\,y\right)\right]\right\} \tag{10}$$

The relation for thermal equilibrium between the surface temperature $\overline{T}_w$
and the ambient temperature T_∞ can be found by evaluating equation
(10) as $y \to \infty$ and is given by

$$\overline{T}_w = T_\infty - \frac{U^2 Pr}{4c_p} \tag{11}$$

Thus, as was anticipated, it can be seen from equation (11) that for
thermal equilibrium the surface must be cooled to compensate for the
aerodynamic heating, and the temperature distribution above a surface
oscillating for a long time in a fluid at temperature T_∞ is given by

$$T_e = T_\infty - \frac{U^2 Pr}{4c_p}\left\{\exp\left(-\sqrt{\frac{2n}{\nu}}\,y\right) - \frac{1}{2-Pr}\left[\exp\left(-\sqrt{\frac{n}{\alpha}}\,y\right)\cos\left(2nt - \sqrt{\frac{n}{\alpha}}\,y\right)\right.\right.$$
$$\left.\left. - \exp\left(-\sqrt{\frac{2n}{\nu}}\,y\right)\cos\left(2nt - \sqrt{\frac{2n}{\nu}}\,y\right)\right]\right\} \tag{12}$$

Now that the fully developed state of the fluid is completely de-
scribed (by eqs. (4) and (12)), consideration can be given to the prob-
lem at hand: namely, the effect of the oscillations on the heat trans-
fer if, at some time (say $t = 0$) after the periodic motion (steady
state) is established, the surface temperature is fixed at some temper-
ature T_w different from that given by equation (11). The problem to be
solved to answer this question is mathematically identical to the one
for thermal equilibrium except that T_w replaces $\overline{T}_w$ in the boundary
conditions and that the solution of the present problem must match that

given by equation (12) for $t = 0$; that is, the proper initial condition must be satisfied by the temperature distribution in order to demonstrate the transient effects properly. It was previously shown that the sum of equations (8) and (9)

$$
\begin{aligned}
T = T_w &+ \frac{U^2 Pr}{4c_p} \left\{ 1 - \exp\left(- \sqrt{\frac{2n}{\nu}}\, y\right) - \mathrm{erf}\left(\frac{y}{2\sqrt{\alpha t}}\right) \right. \\
&+ \frac{1}{2 - Pr} \left[\exp\left(- \sqrt{\frac{n}{\alpha}}\, y\right)\cos\left(2nt - \sqrt{\frac{n}{\alpha}}\, y\right) \right. \\
&\left. \left. - \exp\left(- \sqrt{\frac{2n}{\nu}}\, y\right)\cos\left(2nt - \sqrt{\frac{2n}{\nu}}\, y\right) \right] \right\} \\
&+ (T_\infty - T_w)\,\mathrm{erf}\left(\frac{y}{2\sqrt{\alpha t}}\right)
\end{aligned}
\tag{13}
$$

satisfied the differential equation (eq. (5)) and the boundary conditions (eqs. (6) and (7)), where $\overline{T}_w$ is now replaced by T_w. Evaluation of equations (12) and (13) at $t = 0$ shows that they are identical so that the proper initial condition is satisfied and thus equation (13) represents the temperature distribution over an oscillating surface which is maintained at a constant temperature T_w. Because T_w is a surface temperature different from that for thermal equilibrium, the solution given by equation (13) is of greatest physical significance only as long as the transient terms are important. This of course, is the problem of primary interest herein.

Although the solution given by equation (13) was developed for an incompressible viscous fluid with constant property values, it is equally valid in the case of a compressible viscous fluid if the boundary-layer assumptions are made, if the Prandtl number and the product of ρ and μ are taken to be constants, and if y is replaced by η where

$$
\eta = \int_0^y \frac{\rho}{\rho_\infty}\, dy
$$

This follows because the von Mises transformation (see ref. 6, for example) under these assumptions reduces the compressible boundary-layer equations to the forms of equations (1) and (5).

Any pertinent heat-transfer quantities can be obtained from equation (13). However, since the time dependence itself is usually not of primary practical importance, the temperature gradient at the wall

(which is a measure of the heat transfer) will be averaged over a period. This average gradient is given by

$$\overline{\left(\frac{\partial T}{\partial y}\right)_w} \equiv \frac{n}{2\pi} \int_{t_0}^{t_0 + \frac{2\pi}{n}} \left(\frac{\partial T}{\partial y}\right)_w dt = \frac{U^2 Pr}{2\pi c_p} \sqrt{\frac{n}{2\nu}} \left[\pi - \sqrt{\frac{nPr}{2\pi}} \left(\sqrt{t_0 + \frac{2\pi}{n}} - \sqrt{t_0}\right)\right]$$

$$+ \frac{n(T_\infty - T_w)}{\pi \sqrt{\pi \alpha}} \left(\sqrt{t_0 + \frac{2\pi}{n}} - \sqrt{t_0}\right) \tag{14}$$

where t_0 indicates the beginning of any cycle. For the equilibrium state, an analogous equation is obtained from equation (12):

$$\overline{\left(\frac{\partial T_e}{\partial y}\right)_w} = \frac{U^2 Pr}{2c_p} \sqrt{\frac{n}{2\nu}} \tag{15}$$

This average temperature gradient is due entirely to the aerodynamic heating and is also essentially (that is, except for the multiplicative constant k, the thermal conductivity coefficient) equal to the work done per cycle in oscillating the surface. The latter result can be independently verified by computing the work done per cycle from the shear stress on the plate.

RESULTS AND DISCUSSION

Now that the temperature distribution and its gradient at the surface are known, the effect of the surface oscillations on the heat transfer can be obtained by comparing any appropriate parameters with the corresponding ones for the case of pure conduction to or from a stationary surface subject to the same initial temperature difference $(T_\infty - T_w)$. The specific formulation of the parameters depends on the particular configuration considered. With this in mind, the parameters considered herein will fall into two categories: total parameters, that is, those pertaining to configurations in which the oscillations are inherent; and net parameters for configurations in which the work done in oscillating the surface is assessed to the system. For example, the average total heat-transfer coefficient as given essentially by equation (14) can be compared with that for conduction to or from a stationary surface at an initial temperature difference $(T_\infty - T_w)$. The temperature

distribution for the latter case is given by the sum of the first and last terms of equation (13) (see ref. 5, p. 34) and the associated average temperature gradient is given by merely the last term of equation (14). Therefore, the ratio of these average total heat-transfer coefficients R_1 is given by

$$R_1 = 1 - \left[\frac{U^2 Pr}{4c_p(T_\infty - T_w)} - \frac{U^2 \pi}{2c_p(T_\infty - T_w)} \sqrt{\frac{\pi Pr}{2n}} \frac{1}{\sqrt{t_0 + \frac{2\pi}{n}} - \sqrt{t_0}} \right] \tag{16}$$

It can thus be seen that the oscillations could lead to higher average total heat-transfer coefficients depending on the sign of $(T_\infty - T_w)$ and on the relative magnitude of the two terms in the bracketed part of equation (16); for example, in the limiting case just after the temperature T_w is imposed on the surface, that is, $t_0 \to 0$, equation (16) reduces to

$$R_{10} = 1 - \frac{U^2 \sqrt{Pr}}{4c_p(T_\infty - T_w)} \left(\sqrt{Pr} - \pi \right) \tag{17}$$

so that for the case where the ambient temperature is greater than the surface temperature $(T_\infty > T_w)$, and $Pr < \pi^2$, then $R_{10} > 1$; that is, the oscillations increase the average total heat-transfer coefficient. Conversely if $T_w > T_\infty$, R_{10} would be larger than unity if $Pr > \pi^2$.

It should be reiterated that the parameters for comparison depend upon the configuration considered. Therefore, if in the specific application the net energy obtained from this system (rather than the total as before) is of interest then the total heat transfer must be properly assessed to take account of the work done in oscillating the surface. Hence, the net heat transfer for this case (as represented by the difference between eqs. (14) and (15)) will be compared with that for conduction to or from a stationary surface. The ratio R_2 of the average net heat transfer for the oscillating surface to that for a stationary surface can therefore be written as

$$R_2 \equiv \frac{\overline{\left(\frac{\partial T}{\partial y} \right)}_{w,osc} - \overline{\left(\frac{\partial T_e}{\partial y} \right)}_w}{\overline{\left(\frac{\partial T}{\partial y} \right)}_{w,sta}} = 1 - \frac{U^2 Pr}{4c_p(T_\infty - T_w)} \tag{18}$$

It can be seen from equation (18) that the average net heat-transfer coefficient for the oscillating surface would be larger than that for the stationary surface (that is, $R_2 > 1$) only when $T_w > T_\infty$, and it would

be lower when $T_w < T_\infty$. For a given temperature difference, therefore, the ratio of the average net heat-transfer coefficients, as given by equation (18), can be altered by changing the amplitude of the oscillations.

If the primary interest is not in the net heat transferred across the surface (as it would be, for example, for regenerators) but is in the thermal state of the fluid itself, the effect of the oscillations can be studied by comparing the average static temperature defect defined as

$$\Delta T \equiv \frac{n}{2\pi} \int_{t_0}^{t_0 + \frac{2\pi}{n}} \int_{t_0}^{\infty} (T_\infty - T)\, dy\, dt \qquad (19)$$

for the oscillating surface to that for the stationary surface. The ratio of these defects is given by

$$R_3 \equiv \frac{\Delta T_{osc}}{\Delta T_{sta}} = 1 - \frac{U^2 Pr}{4 c_p (T_\infty - T_w)} + \frac{3\pi U^2 Pr}{8 n c_p (T_\infty - T_w)} \sqrt{\frac{Pr\pi}{2n}} \left[\left(t_0 + \frac{2\pi}{n} \right)^{\frac{3}{2}} - \left(t_0 \right)^{\frac{3}{2}} \right]^{-1}$$

$$(20)$$

For the limiting case, $t_0 \to 0$, equation (19) reduces to

$$R_{30} = 1 - \frac{U^2 Pr}{4 c_p (T_\infty - T_w)} \left(1 - \frac{3}{8} \sqrt{Pr} \right) \qquad (21)$$

It can thus be seen that unless $Pr > 64/9$ the oscillations will not be advantageous for lowering the fluid temperature for the case when $T_\infty > T_w$.

A more meaningful parameter to examine would perhaps be the ratio of the average total-temperature defect with the oscillations to that for conduction to the stationary surface. The average total-temperature defect is defined as in equation (19) except that T is replaced by the total temperature T_t, which is given by

$$T_t = T + \frac{u^2}{2 c_p}$$

In this way the thermal condition of the fluid independent of its motion is indicated. This ratio is given by

$$R_4 = 1 - \frac{U^2 Pr}{4c_p(T_\bullet - T_w)} \left\{ 1 + \frac{3\pi(1 - Pr)}{2n} \sqrt{\frac{\pi}{2nPr}} \left[\frac{1}{\left(t_0 + \frac{2\pi}{n}\right)^{\frac{3}{2}} - \left(t_0\right)^{\frac{3}{2}}} \right] \right\}$$

(22)

Thus, the ratio R_4 could be greater than unity dependent on the relative orders of magnitude of the terms in the braces in equation (22) and in the limiting case as $t_0 \to 0$.

$$R_{40} = 1 - \frac{U^2 Pr}{4c_p(T_\bullet - T_w)} \left[1 + \frac{3}{8} \frac{(1 - Pr)}{\sqrt{Pr}} \right]$$

(23)

Thus, in this case if $T_\bullet > T_w$ the fluid would be at a lower relative temperature with the oscillations $(R_{40} > 1)$ only if $Pr > 9$.

An analogous discussion, of course, follows for increasing the fluid temperature.

CONCLUDING REMARKS

The results of this preliminary analysis of the temperature distribution over conducting oscillating surfaces have shown that the heat transfer associated with the oscillating surface can be significantly different from that for conduction between the same initial temperature difference and a stationary surface. In addition to the heat transfer, the thermal state of the fluid was studied by means of average static- or total-temperature defects and it was found that the surface oscillations can alter the thermal state of the fluid. In each case, the range of parametric values for which the surface oscillations are beneficial were indicated.

Lewis Flight Propulsion Laboratory
    National Advisory Committee for Aeronautics
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APPENDIX - SYMBOLS

The following symbols are used in this report:

c_p specific heat at constant pressure

$\text{erf}(\beta) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^\beta \exp(-r^2)\,dr$

k thermal conductivity coefficient

n frequency of oscillations

Pr Prandtl number

R_1 ratio of average total heat-transfer coefficients

R_{10} ratio of average total heat-transfer coefficients evaluated as $t_0 \to 0$

R_2 ratio of average net heat-transfer coefficients

R_3 ratio of average static-temperature defects

R_{30} ratio of average static-temperature defects evaluated as $t_0 \to 0$

R_4 ratio of average total-temperature defects

R_{40} ratio of average total-temperature defects evaluated as $t_0 \to 0$

T static temperature

T_e equilibrium (or steady-state) static temperature

T_t total temperature

T_w surface temperature

$\overline{T}_w$ surface temperature for steady state

T_∞ static temperature in ambient fluid

ΔT temperature defect

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t	time
t_0	time at the start of an arbitrary cycle
U	amplitude of oscillations
u	velocity parallel to surface
y	coordinate normal to surface
α	thermal diffusivity
η	von Mises coordinate normal to surface
μ	absolute viscosity
ν	kinematic viscosity
ρ	density

Subscripts:

c	complementary solution
osc	oscillating surface
p	particular solution
sta	stationary surface

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